STRONG MINIMAL COVERS AND A QUESTION OF YATES: THE STORY SO FAR

ANDREW E.M. LEWIS

Abstract. An old question of Yates as to whether all minimal degrees have a strong minimal cover remains one of the longstanding problems of degree theory, apparently largely impervious to present techniques. We survey existing results in this area, focussing especially on some recent progress.

1. Introduction

By the 60's and 70's degree theorists had become concerned with some particular and fundamental questions of a global nature concerning the structure of the Turing degrees. In order to address issues regarding homogeneity and the decidability and degree of the theory, the approach taken at this time was to proceed through a deep analysis of the initial segments of the structure. Along these lines a technique for piecemeal construction of initial segments, even if only locally successful, would have been very useful and it was in this context that interest was first aroused in a question of Yates:

Definition 1.1. *A degree* $\mathbf{b}$ *is a strong minimal cover for* $\boldsymbol{a}$ *if* $\mathcal{D}[<\boldsymbol{b}]=\mathcal{D}[\leq \boldsymbol{a}]$. *A degree* $\mathbf{a}$ *is minimal if it is a strong minimal cover for* $\mathbf{0}$.

Question 1.1. *(Yates) Does every minimal degree have a strong minimal cover?*

In fact, the question of characterizing those degrees with strong minimal cover had already been raised by Spector in his 1956 paper [CS]. Certainly in $\mathcal{D}_m$—the structure of the many-one degrees, induced by a strengthening of the Turing reducibility—Lachlan's proof of the fact that every m-degree has a strong minimal cover played a vital role in Ershov's [YE] and Paliutin's [EP] results characterizing the structure and in showing, for instance, that $\mathbf{0}_m$ is the only definable singleton. In the Turing degrees, however, progress with global concerns of this nature was eventually achieved through other means—initially through complicated ad hoc initial segment embeddings, and more recently using simpler coding techniques. Yates' question remained behind, apparently largely impervious to existing techniques.

At the opposite end of the spectrum to those degrees with a strong minimal cover are those which satisfy the cupping property, where we say that a degree $\mathbf{a}$ satisfies the cupping property if for every $\mathbf{b}>\mathbf{a}$ there exists

2000MSC: 03D28. The author was supported by Marie-Curie Fellowship MEIF-CT-2005-023657 and was partially supported by the NSFC Grand International Joint Project no. 60310213, New Directions in the Theory and Applications of Models of Computation.

$\mathbf{c} < \mathbf{b}$ with $\mathbf{a} \vee \mathbf{c} = \mathbf{b}$. We say that a degree $\mathbf{a}$ is generalized low$_n$ (GL$_n$) if $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n-1)}$. Since Jockusch and Posner have shown that all degrees which are not GL$_2$ satisfy the cupping property, possession of a strong minimal cover may be seen as a property satisfied by degrees which are in some sense *low down* in the structure. Until very recently, however, results on the positive side of Yates' question were restricted to exhibiting particular examples of individual degrees which possess a strong minimal cover. As well as initial segment results along these lines, Cooper [BC3] showed that there exists a non-zero c.e. degree with a strong minimal cover—the motivation at that time being an approach towards the definability of $\mathbf{0}'$. Much later Kumabe [MK2] was able to show that there exists a 1-generic degree with this property.

It was not until 1999 that Ishmukhametov [SI] was able to provide an interesting *class* of degrees—the c.e. traceable degrees— every member of which has a strong minimal cover. In sections 4 and 5 we shall provide an alternative proof of Ishmukhametov's result, appearing for the first time in this paper, by showing that the c.e. traceable degrees are actually a proper subclass of another very natural class of degrees all of which have strong minimal cover. In section 3 we shall sketch an alternative proof of what may be regarded the strongest result on the negative side of Yates' question, the fact that there exists a hyperimmune-free degree with no strong minimal cover. Finally in section 5 we shall go on to discuss some other results recently obtained by the author in this area. In order to do so, it is necessary that we should begin with some discussion of the splitting tree technique of minimal degree construction.

2. The splitting tree technique and strong minimal covers

The splitting tree technique is very much the standard approach to minimal degree construction. In fact really it is more than that—almost without exception, when a computability theorist wishes to construct a set of minimal degree it is the splitting tree technique that they will use. For a clear introduction to the techniques of minimal degree construction we refer the reader to any one of [RS], [BC2], [ML], [PO1]. This paper requires knowledge only of those techniques for constructing a minimal degree below $\mathbf{0}''$.

Given any $T \subseteq 2^{<\omega}$ and $\tau \in T$, we shall say that τ is a leaf of T if it has no proper extensions in T. When $\tau, \tau' \in T$ and $\tau \subset \tau'$ we call τ' a successor of τ in T if there doesn't exist $\tau'' \in T$ with $\tau \subset \tau'' \subset \tau'$. Given any $T \subseteq 2^{<\omega}$ and $A \subseteq \omega$ we denote $A \in [T]$ and we say that A lies on T, if there exist infinitely many initial segments of A in T. We shall say that $\tau \in T$ is of level n in T if τ has precisely n proper initial segments in T and we shall say that finite T is of level n if all leaves of T are of level n in T. The following definition is slightly non-standard, but seems convenient where the discussion of splitting trees with unbounded branching is concerned: we say that $T \subseteq 2^{<\omega}$ is a c.e. tree if it has a computable enumeration $\{T_s\}_{s\geq 0}$ such that $|T_0| = 1$ and such that for all $s \geq 0$ if $\tau \in T_{s+1} - T_s$ then τ extends a leaf of T_s (and such that a finite number of strings are enumerated at any given stage). For any Turing functional Ψ, we say that two finite strings τ

and τ' are a Ψ-splitting if $\Psi(\tau)$ and $\Psi(\tau')$ are incompatible. We say that $T \subseteq 2^{<\omega}$ is Ψ-splitting if whenever $\tau, \tau' \in T$ are incompatible these strings are a Ψ-splitting. We shall say that τ is $A\oplus$-compatible if, for all n such that $\tau(2n)\downarrow$, we have $\tau(2n) = A(n)$. For any $\tau \in 2^{<\omega}$ if $|\tau| > 0$ we define τ^- to be the initial segment of τ of length $|\tau| - 1$, and otherwise we define $\tau^- = \tau$. Given any Turing functional Ψ we define $\hat{\Psi}$ as follows. For all τ and all n, $\hat{\Psi}(\tau; n) \downarrow = x$ iff the computation $\Psi(\tau; n)$ converges in $<|\tau|$ steps, $\Psi(\tau; n) = x$ and $\hat{\Psi}(\tau^-; n') \downarrow$ for all $n' < n$. We say that $T \subseteq 2^{<\omega}$ with a single string of level 0 is 2-branching if every member of T has precisely two successors. We say non-empty $T \subseteq 2^{<\omega}$ is perfect if each $\tau \in T$ has at least two successors.

The basic problem which presents itself in attempting to construct a strong minimal cover for any given degree **a** is that if we simply relativize the standard minimal degree construction to **a** then, since the splitting trees concerned will now only be c.e. in **a**, what results will certainly be a minimal cover for **a** but will not necessarily be a strong minimal cover—and where we say that **b** is a minimal cover for **a** if **b** > **a** and there does not exist **c** with **a** < **c** < **b**. As was observed in [AL1], however, a simple analysis of this construction can be used in order to provide a class of degrees for which the construction of a strong minimal cover is possible.

Lemma 2.1. *Suppose $\Psi = \hat{\Psi}$. If T_0 is A-computable, 2-branching and Ψ-splitting, then $T_1 = \{\Psi(\tau) : \tau \in T_0\}$ is A-computable and 2-branching. Let $T_2 \subseteq T_1$ be A-computable and 2-branching. Then $T_3 = \{\tau \in T_0 : \Psi(\tau) \in T_2\}$ is A-computable and 2-branching with $T_3 \subseteq T_0$.*

Proof. The proof is not difficult and is left to the reader. □

Now let $\{\Psi_i\}_{i \geq 0}$ be some fixed effective listing of the Turing functionals and let us suppose that we wish to construct a strong minimal cover for **a**. In order to do so we suppose given $A \in$ **a** and we must construct $B \geq_T A$ so as to satisfy all requirements;

$\mathcal{R}_i : \Psi_i(B)$ total $\rightarrow (\Psi_i(B) \leq_T A$ or $B \leq_T \Psi_i(B))$
$\mathcal{P}_i : B \neq \Psi_i(A)$

A standard technique can be used in order to ensure that $B \geq_T A$, we simply insist that B should be an $A\oplus$-compatible string. Thus we begin with the restriction that B should lie on T_0 which contains all strings of even length which are $A\oplus$-compatible. T_0, then, is A-computable and 2-branching.

In order to meet all other requirements we might try to proceed with an easy forcing argument. We define B_0 to be the empty string. Suppose that by the end of stage s we have defined $B_s \in 2^{<\omega}$ on T_s, which is A-computable and 2-branching, in such a way that if B extends B_s and lies on T_s then all requirements $\mathcal{R}_i, \mathcal{P}_i$ for $i < s$ will be satisfied. At stage $s+1$ we might proceed, initially, just as if we were only trying to construct a minimal cover for **a**. We can assume that $\Psi_s = \hat{\Psi}_s$, otherwise replace Ψ_s with $\hat{\Psi}_s$ in what follows. We ask the question, "does there exist $\tau \supseteq B_s$ on T_s such that no two strings on T_s extending τ are a Ψ_s-splitting?".

If so: then let τ be such a string. We can define $T_{s+1} = T_s$ and (just to make the satisfaction of $\mathcal{P}_s$ explicit) define B_{s+1} to be some extension of τ on T_s sufficient to ensure $\mathcal{P}_s$ is satisfied.

If not: then we can define T'_s to be an A-computable 2-branching Ψ_s-splitting subset of T_s having B_s as least element—the idea being that we shall eventually define T_{s+1} to be some subset of T'_s. If B lies on T'_s then we shall have that $B \leq_T \Psi_s(B) \oplus A$. Of course this does not suffice, since for the satisfaction of $\mathcal{R}_s$ we require that $B \leq_T \Psi_s(B)$. Suppose, however, that we know **a** is a *tree basis*:

Definition 2.1. *We say deg(A) is a tree basis if for every perfect T computable in A there exists some perfect $T' \subseteq T$ computable in A such that every $C \in [T']$ computes A.*

Lemma 2.1 then suffices to ensure that we can define T_{s+1} to be a subset of T'_s which is A-computable and 2-branching, and which satisfies the property that if $B \in [T_{s+1}]$ then $A \leq_T \Psi_s(B)$ so that, since $B \leq_T \Psi_s(B) \oplus A$, $B \leq_T \Psi_s(B)$. Then we can define B_{s+1} to be an extension of B_s lying on T_{s+1} sufficient to ensure the satisfaction of $\mathcal{P}_s$.

We therefore have:

Theorem 2.1. *(Lewis* [AL1]*) Every tree basis has a strong minimal cover.*

2.1. **Delayed splitting trees.** Given the monopoly that the splitting tree technique presently has on minimal degree construction it seems an obvious question to ask the generality of this technique. Since it seems to be in the very least difficult to construct a negative solution to Yates' question using the splitting tree technique, it is a natural question to ask whether we should be looking for other techniques of minimal degree construction. So perhaps it is interesting to observe that in order to obtain a perfectly general technique of minimal degree construction all one need do is to use splitting trees with splitting delayed by one level at a time:

Definition 2.2. *We say that $T \subseteq 2^{<\omega}$ is a delayed Ψ-splitting c.e. tree if T is a c.e. tree and whenever $\tau_0, \tau_1 \in T$ are incompatible any $\tau_2, \tau_3 \in T$ properly extending τ_0 and τ_1 respectively are a Ψ-splitting.*

Theorem 2.2. *(Lewis* [AL1]*) A is of minimal degree iff A is noncomputable and, whenever $\Psi(A)$ is total and noncomputable, A lies on a delayed Ψ-splitting c.e. tree.*

Delayed splitting trees, then, provide a perfectly general technique of minimal degree construction in the sense that any set of minimal degree lies on trees of this kind. To put it another way, a trivial modification of the proof of theorem 2.2 suffices to show that any set of minimal degree is generic for the associated notion of forcing. One might ask, however, whether we really need to use delayed splitting trees in order to get this result, perhaps it already holds for standard splitting trees? In fact this is not the case. In order to see this we need to consider the fixed point free degrees:

Definition 2.3. *Let ϕ_n denote the n^{th} partial computable function according to some fixed effective listing. $A \subseteq \omega$ is of fixed point free (FPF) degree if there exists $f \leq_T A$ with $\phi_n \neq \phi_{f(n)}$ for all n.*

It was another reasonably longstanding question regarding minimal degrees as to whether there exists a minimal degree which is FPF. Kumabe was able to answer this question in the affirmative:

Theorem 2.3. *(Kumabe* [MK1]*) There exists a FPF minimal degree.*

In order to achieve this result Kumabe uses delayed splitting trees—in fact this is the only example in the literature of such a construction—and one can show that it is necessary that he should do so:

Theorem 2.4. *(Lewis* [AL1]*) If A lies on a Ψ-splitting c.e. tree whenever $\Psi(A)$ is total and noncomputable, then A is not of FPF degree.*

3. A hyperimmune-free degree with no strong minimal cover

Recall that A is of hyperimmune-free degree if for every $f \leq_T A$ there exists computable g which majorizes f i.e. such that $g(n) > f(n)$ for all n, and that a degree is PA if it computes a complete extension of Peano Arithmetic. In some ways the sets of hyperimmune-free degree and the sets of minimal degree can be seen as being very closely related, at least in the sense that the standard constructions are very similar. In fact the most primitive form of minimal degree construction uses perfect splitting trees and it is not difficult to show that if A satisfies the property that whenever $\Psi(A)$ is total and noncomputable A lies on a perfect c.e. Ψ-splitting tree, then A will automatically be of hyperimmune-free degree. Given this close relationship, perhaps the strongest result on the negative side of Yates' question is the fact that there exists a hyperimmune-free degree with no strong minimal cover.

Theorem 3.1. *(Kucera* [AK]*) Every PA degree satisfies the cupping property.*

Corollary 3.1. *There exists a hyperimmune-free degree with no strong minimal cover.*

Proof. There exists a PA degree which is hyperimmune-free. □

Definition 3.1. *We say that $\Lambda \subseteq 2^{<\omega}$ is downward closed if whenever $\tau \in \Lambda$ and $\tau' \subset \tau$ we have $\tau' \in \Lambda$. $\mathcal{P} \subseteq 2^{\omega}$ is a Π^0_1 class if it is $[\Lambda]$ for some downward closed computable Λ.*

Alternative proof of theorem 3.1 (Sketch). In [AL1] the author described an alternative proof of theorem 3.1 which does not require reasoning within PA like the original and which has subsequently turned out to be useful in proving other results. It is this alternative proof which we shall sketch here.

In fact, the statement of the theorem is easily seen to be equivalent to the following proposition: there exists a non-empty Π^0_1 class every member of which is of degree which satisfies the cupping property. In one direction this follows immediately because there exists a non-empty Π^0_1 class which contains only sets which effectively code a complete extension of PA. For the other direction it suffices to recall that A is of PA degree iff A computes a member of every non-empty Π^0_1 class and then to observe that the degrees which satisfy the cupping property form an upward closed class. It is this equivalent form of the theorem which we shall prove. In order to do so we make use of the following very simple lemma:

Lemma 3.1. *If A computes 2-branching T such that no set lying on T computes A, then the degree of A satisfies the cupping property.*

Proof. Suppose that A and T satisfy the hypothesis of the lemma and that we are given $B >_T A$. We define $C = \bigcup_s \tau_s$ such that $C <_T B$ and $B \leq_T C \oplus A$. We define τ_0 to be the string of level 0 in T. Given τ_s we define τ_{s+1} to be the left successor of τ_s in T if $B(s) = 0$ and the right successor otherwise. Then C lies on T so that $C <_T B$, since no set lying on T computes A. If we are given an oracle for A and an oracle for C then clearly we can compute B. □

What we aim to achieve, then, is to construct downward closed computable Λ such that there exist infinite paths through Λ and such that if A is an infinite path through Λ then A computes some non-empty 2-branching T^A (let's say) such that no set lying on T^A computes A. In order to ensure that no set lying on T^A computes A it is convenient to construct a Turing functional Ψ such that no set on T^A computes $\Psi(A)$. In order to define T^A for any A which is an infinite path through Λ we shall define values T^τ for τ in Λ and then T^A will be defined to be the union of all T^τ such that $\tau \subset A$. Thus there are three different kinds of object under construction, Λ, T^A for A which is an infinite path through Λ and Ψ, and we must define these values in such a way that there exist infinite paths through Λ and so that the following requirements are satisfied:

$$\mathcal{N}_i : (A \in [\Lambda] \wedge C \in [T^A]) \rightarrow (\Psi_i(C; i) \neq \Psi(A; i)).$$

In fact, what we shall do here is just to consider how to satisfy a single requirement $\mathcal{N}_0$. We shall therefore only be concerned with the values $\Psi_0(C; 0)$ and with defining Ψ on argument 0.

The most primitive form of the intuition runs as follows: if we are given four strings and we colour those four strings using two colours then there exists some colour such that at least two strings are not that colour (okay so actually we only need three strings but it is convenient here to do everything in powers of two). Now we extend this idea. First we define a certain set of strings T. The role of T is that it is the set of all strings that could possibly be in T^τ for some $\tau \in \Lambda$. In the case that we are only looking to satisfy a single requirement T is rather trivial, we just define T to be the set of all strings of even length. The important thing about T is that it is 4-branching. We let $T(n)$ denote the set of strings in T of level $\leq n$ in T. Next, we consider a certain form of finite subset of T:

Definition 3.2. *We say that finite $T' \subset T$ is $(T, 2)$-compatible if the strings of level n in T' are of length $2n$ and every string in T' which is not a leaf of T' has precisely two successors.*

The role of these $(T, 2)$-compatible T' is that when we define T^τ for $\tau \in \Lambda$ actually we shall define this value to be some $(T, 2)$-compatible T'. The following lemma is what we need in order to satisfy the first requirement:

Lemma 3.2. *For any finite $T' \subseteq 2^{<\omega}$ a 2-colouring of T' is an assignment of some $col(\sigma) \in \{0,1\}$ to each leaf σ of T'. For any n and any 2-colouring of $T(n)$ there exists T' which is $(T,2)$ compatible of level n and $d \in \{0,1\}$ such that no leaf σ of T' has $col(\sigma) = d$.*

Proof. This is easily seen by induction. The case $n = 0$ is trivial and, in fact, we have already seen the case $n = 1$. If we are given four strings and we colour those four strings using two colours then there exists some colour such that at least two strings are not that colour. Those two strings then define some $(T,2)$-compatible T' of level 1. In order to see the induction step suppose we are given a 2-colouring of $T(n+1)$. First we use this 2-colouring in order to define a 2-colouring of $T(n)$ as follows. Consider each leaf σ of $T(n)$. Such σ has precisely four successors in $T(n+1)$. If more than two of those successors are coloured 0 then colour σ with 0. If more than two of those successors are coloured 1 then colour σ with 1, and otherwise colour σ with 0. What this means is that if σ is not coloured d then at least two successors of σ are not coloured d. By the induction hypothesis there exists some $(T,2)$-compatible T' of level n and there exists d such that no leaf of T' is coloured d. In order to define T'' of level $n+1$ sufficient to complete the induction step all we need do is to choose two successors of each leaf of T' which are not coloured d. □

Now we see how to use this lemma in order to satisfy $\mathcal{N}_0$ while also satisfying the condition that $[\Lambda]$ should be non-empty. Before defining Λ we define a set of strings $\Lambda^\star$. These are strings which may or may not be in Λ. We do not insist that $\Lambda^\star$ should be downward closed, in order to form Λ we shall later add strings in so that Λ is downward closed. For every n we let $\Lambda^\star(n)$ denote the set of strings in $\Lambda^\star$ which are of level n in $\Lambda^\star$. Thus for every n we must define the set of strings which are in $\Lambda^\star(n)$, for each such τ we must define a value T^τ which will be some $(T,2)$-compatible T' of level n and we must also ensure that if $n > 0$ then $\Psi(\tau)$ is defined on argument 0. In order to satisfy this latter condition we can just ensure that $\Psi(\tau;0)$ is defined for all τ of level 1 in $\Lambda^\star$ and then this task is done once and for all. We shall not describe here precisely how to define these values, but hopefully it is clear that we can do so in such a way that the following lemma is satisfied:

Lemma 3.3. *For any $n > 0$, any $(T,2)$-compatible T' of level n and any $d \in \{0,1\}$ there exists $\tau \in \Lambda^\star(n)$ such that $T^\tau = T'$ and $\Psi(\tau;0) = d$.*

The fact that we can satisfy lemma 3.3 is really completely obvious. We are not insisting that $\Lambda^\star$ should be downward closed, so in order to ensure satisfaction of the lemma all we need do is to put enough strings into each $\Lambda^\star(n)$ so that all possibilities can be realised.

What this means is that if we define Λ by taking the strings in $\Lambda^\star$, adding in strings in order to make it downward closed and then removing any string τ (together with all extensions) for which it is the case that there exists $\sigma \in T^\tau$ with $\hat{\Psi}_0(\sigma;0) \downarrow = \Psi(\tau;0)$ then for every n we must be left with strings in $\Lambda^\star(n)$ which are in Λ. In order to see this suppose given $n > 0$. Then we can consider the values $\hat{\Psi}_0(\sigma;0)$ for those $\sigma \in T(n)$ to define a 2-colouring of $T(n)$—where values are not defined to be either 0 or 1 we need not be concerned with them. Then lemma 3.2 tells us that for this 2-colouring of $T(n)$ there exists some $(T, 2)$-compatible T' of level n and there exists $d \in \{0, 1\}$ such that no leaf of T' is coloured d. Fixing such T' and such d we may then apply lemma 3.3 which tells us that there exists $\tau \in \Lambda^\star(n)$ with $T^\tau = T'$ and $\Psi(\tau; 0) = d$. This string τ, then, is a string in $\Lambda^\star(n)$ which is in Λ. By weak König's lemma the fact that there exist an infinite number of strings in Λ suffices to ensure that $[\Lambda]$ is nonempty.

In order to satisfy all requirements we must become a little more sophisticated —we need more colours and bushier trees—but the basic idea remains the same.

Upon seeing this result it seems a natural question to ask what we can say (to the opposite extreme) about Π^0_1 classes every member of which is of degree with a strong minimal cover. Of course it is a trivial matter to construct a non-empty class of this kind because all we need do is to include a single computable member. On the other hand we cannot hope to construct such a class of positive measure because any such class contains a member of every degree above $\mathbf{0}'$ and so certainly contains members of degree with no strong minimal cover. In a sense, then, perhaps the following result is the strongest we could hope for:

Theorem 3.2. *(Lewis* [AL1]*) There exists a non-empty Π^0_1 class with no computable members, every member of which is of degree with a strong minimal cover.*

4. The c.e. traceable degrees

As was outlined in the introduction, it seems fair to say that for a long time very little progress was made in the attempt to understand the issues surrounding Yates' problem. To a certain extent this changed relatively recently with Ishmukhametov's characterization of the c.e. degrees which have a strong minimal cover. In order to obtain this characterization Ishmukhametov built on previous work by Downey, Jockusch and Stob, who considered the array noncomputable degrees.

Definition 4.1. *(Downey, Jockusch, Stob* [DJS]*) A degree* $\mathbf{a}$ *is array noncomputable (a.n.c.) if for each $f \leq_{wtt} K$ there is a function g computable in* $\mathbf{a}$ *which is not dominated by f i.e. such that $g(n) \geq f(n)$ for infinitely many n.*

Here f being *wtt* reducible to K just means that f is computable in the halting problem and that there is a computable bound on the *use* of the computation i.e. the number of bits of the oracle tape scanned on any given argument. It is an interesting characteristic of the a.n.c. degrees that they satisfy many of the properties of the degrees which are not generalized low$_2$. The following result, in particular, is the one that is of relevance to us now:

Theorem 4.1. *(Downey,Jockusch,Stob* [DJS]*) Given* $\mathbf{a}$ *which is a.n.c.:*

(1) $\mathbf{a}$ *is not minimal,*
(2) $\mathbf{a}$ *satisfies the cupping property.*

In fact, it is not difficult to provide an alternative proof of theorem 4.1 by showing that the a.n.c. degrees are a proper subclass of those degrees whose sets satisfy the hypothesis of lemma 3.1:

Theorem 4.2. *The a.n.c. degrees are a proper subclass of those degrees* **a** *satisfying* (†) *if* $A \in \mathbf{a}$ *then* A *computes 2-branching* T *such that no set lying on* T *computes* A.

Proof. The proof of theorem 3.1 described in [AL1] and sketched in the last section suffices to show that there exists a hyperimmune-free degree which satisfies (†). Since no hyperimmune-free degree can be a.n.c. we are left to show that any a.n.c. degree satisfies (†).

So suppose given A of a.n.c. degree. For any τ we define $f(\tau)$ as follows. For each $i \leq \mid \tau\mid$ let s_i be the least such that there exist τ_0, τ_1 extending τ of length s_i and which are a $\hat{\Psi}_i$-splitting and if there exist no such τ_0, τ_1 then let $s_i = 0$. Define $f(\tau) = \max\{s_i : i \leq \mid \tau\mid\}$. For every s define $f^\star(s) = \max\{f(\tau) :\mid \tau\mid = s\}$. Then $f^\star \leq_{wtt} K$ and since the degree of A is a.n.c. there exists $g \leq_T A$ with $g(s) \geq f^\star(s)$ for infinitely many s. We can assume that g is an increasing function.

We define $T = \bigcup_s T_s$ as follows.

Stage 0. We define T_0 to be $\{\lambda\}$ (and where λ is the string of length 0).
Stage $s+1$. For each leaf τ of T_s and each $i \leq \mid \tau \mid$ such that $\hat{\Psi}_i(\tau)$ is compatible with A, check to see whether there exists a $\hat{\Psi}_i$-splitting τ_0, τ_1 such that both these strings extend τ and are of length $\leq g(s)$.
If so (for some $i \leq \mid \tau\mid$): then let i be the least such, let $\tau' \supset \tau$ be as short as possible such that $\hat{\Psi}(\tau')$ is incompatible with A and if $\mid\tau'\mid \leq s$ then enumerate the two one element extensions of τ' into T_{s+1}.
If not: then enumerate the two one element extensions of τ into T_{s+1}.

The verification that T is 2-branching and that no set lying on T computes A is not difficult and is left to the reader. □

In order to show that theorem 4.1 follows from theorem 4.2 it remains to show that no a.n.c. degree is minimal. So suppose towards a contradiction that A is of minimal a.n.c. degree **a**. Then clearly **a** is hyperimmune, so we can take increasing $f \leq_T A$ which is not dominated by any computable function. By theorem 4.2 we can suppose given $T \leq_T A$ which is 2-branching and such that no set lying on T computes A. We define $C \leq_T A$ with $C \in [T]$ which is noncomputable. Since A is of minimal degree this suffices to ensure that $A \leq_T C$, which gives the required contradiction.

Let ϕ_i be the i^{th} partial computable function according to some fixed effective listing. The e-state of τ at stage s is the binary string σ of length e defined as follows: for all $i < e$, $\sigma(i) = 1$ iff $\phi_i[s]$ is compatible with τ—and where $\phi_i[s]$ is the approximation to ϕ_i at stage s. For all s, $h(s)$ is defined to be the length of the longest string of level $s+1$ in T. We define $C = \bigcup_s \tau_s$ as follows.

Stage 0. Define τ_0 to be the string of level 0 in T.

Stage $s+1$. Define τ_{s+1} to be the successor of τ_s in T which has the lexicographically least $(s+1)$-state at stage $f(h(s+1))$ (or to be the leftmost if both successors have the same $(s+1)$-state at stage $f(h(s+1))$).

To prove that C is noncomputable is not difficult and is left to the reader.

Ishmukhametov then combined theorem 4.1 to great effect with work on c.e. traceability:

Definition 4.2. *$A \subseteq \omega$ is c.e. traceable if there is a computable function p such that for every function $f \leq_T A$ there is a computable function h such that $|W_{h(n)}| \leq p(n)$ and $f(n) \in W_{h(n)}$ for all $n \in \omega$.*

How should one understand this definition? The function p here can be thought of as a *bounding* function and the function h can be thought of as a *guessing* function. Thus A is c.e. traceable if there exists some computable bounding function p such that for every $f \leq_T A$ there exists some computable guessing function h which makes at most $p(n)$ guesses as regards each value $f(n)$ and one of these guesses is always correct. Having observed that the class of c.e. traceable degrees (those whose sets are c.e. traceable) is complementary to the class of a.n.c. degrees in the c.e. degrees, Ishmukhametov [SI] was then able to show:

Theorem 4.3. *(Ishmukhametov* [SI]*) All c.e. traceable degrees have a strong minimal cover.*

Corollary 4.1. *A c.e. degree has a strong minimal cover iff it is c.e. traceable.*

In fact, given theorem 2.1, theorem 4.3 follows as an immediate corollary of theorem 4.4 below. In the next section we shall be able to observe that the c.e. traceable degrees are actually a proper subclass of those degrees which are a tree basis.

Theorem 4.4. *Every c.e. traceable degree is a tree basis.*

Proof. Terwijn and Zambella [TZ] have shown that being c.e. traceable is actually equivalent to satisfaction of the following condition:

($\dagger_0$) For every function $f \leq_T A$ there is a computable function h such that $|W_{h(n)}| \leq 2^n$ and $f(n) \in W_{h(n)}$ for all $n \in \omega$.

It is this characterization of the c.e. traceable sets that we shall use in what follows. For any finite T' we shall say that T' is 2-branching to level n if it is of level n and every string in T' which is not a leaf of T' has precisely two successors. In what follows we shall subdue mention of effective codings between strings and natural numbers and between finite sets of strings and natural numbers for the sake of readability. So now assume that A is c.e. traceable, $T \leq_T A$ and that T is perfect. In fact we may assume without loss of generality that T is 2-branching and that we are given Ψ such that:

- for any τ, the value $\Psi(\tau)$ is 2-branching to level n for some n and can be computed in $|\tau|$ many steps,
- $\Psi(A) = T$,
- for any $\tau \subset \tau'$, if $\sigma \in \Psi(\tau)$ then a) $\sigma \in \Psi(\tau')$ and b) if $\sigma' \subset \sigma$ and $\sigma' \notin \Psi(\tau)$ then $\sigma' \notin \Psi(\tau')$,
- if $\Psi(\tau)$ is of level $n > 0$ then there exists $\tau' \subset \tau$ such that $\Psi(\tau')$ is of level $n - 1$.

First we define the function $f \leq_T A$ as follows; for every n, $f(n)$ is the shortest initial segment of A, τ say, such that $\Psi(\tau)$ is of level $\Sigma_{i=0}^n(i+2)$. Since A is c.e. traceable we can then take computable h such that $|W_{h(n)}| \leq 2^n$ and $f(n) \in W_{h(n)}$ for all $n \in \omega$. We can assume that if $n > 0$ then τ is not enumerated into $W_{h(n)}$ until some initial segment of τ has been enumerated into $W_{h(n-1)}$ and unless $\Psi(\tau)$ is of level $\Sigma_{i=0}^n(i+2)$ and this is not the case for any proper initial segment of τ.

Next we proceed to computably enumerate various values $T_0(\tau)$ and $T_1(\tau)$ and axioms for a Turing functional Φ such that $T_0(A)$ is 2-branching and for every set C lying on $T_0(A)$ we have $\Phi(C) = A$. Initially all strings are available for use. There can be only a single string enumerated into $W_{h(0)}$, τ say. We define $T_1(\tau)$ to be the strings of level 0 and 2 in $\Psi(\tau)$ and we define $T_0(\tau)$ to be the string, σ say, of level 0 in $\Psi(\tau)$. We declare σ to be unavailable for use and enumerate the axiom $\Phi(\sigma) = \tau$. Whenever some τ is enumerated into $W_{h(n)}$ for $n > 0$ we shall already have enumerated a (unique and proper) initial segment of this string, τ' say, into $W_{h(n-1)}$. For each leaf σ of $T_0(\tau')$ there will be precisely 2^{n+1} successors in $T_1(\tau')$. Let σ_0 and σ_1 be the first two of these which are still available for use, enumerate these strings into $T_0(\tau)$ and enumerate all leaves σ' of $\Psi(\tau)$ which extend these two strings into $T_1(\tau)$ before enumerating the axioms $\Phi(\sigma_0) = \tau$, $\Phi(\sigma_1) = \tau$. Declare σ_0 and σ_1 to be unavailable for use.

In order to see that the axioms enumerated for Φ are consistent observe that when we enumerate an axiom $\Phi(\sigma) = \tau$, σ is a leaf of $T_0(\tau)$. The consistency of the axioms enumerated for Φ then follows from the fact that it is easily seen by induction on the stage of the construction that if τ and τ' are incompatible and we have defined values $T_0(\tau), T_0(\tau')$ then the leaves of these two sets of strings are pairwise incompatible. The fact that $T_0(A)$ is 2-branching and that for every $C \in T_0(A)$ we have $\Phi(C) = A$ then follows immediately from the description of the construction.

□

Unfortunately c.e. traceability does not relate in such a tidy way where the minimal degrees are concerned. Gabay [YG] has shown that there exist minimal degrees which are not c.e. traceable and which have a strong minimal cover and that there exist minimal degrees below $\mathbf{0}'$ which are not c.e. traceable. Theorem 4.3 does suffice to show, however, that many of the minimal degrees which we typically construct will automatically have a strong minimal cover. Let us say that a splitting tree has bounded branching if there exists n such that each string in the tree has at most n successors. It is not difficult to show that if A satisfies the property that whenever $\Psi(A)$ is total and noncomputable A lies on a c.e. Ψ-splitting tree with bounded branching, then A is c.e. traceable. The vast majority of minimal degree constructions in the literature use splitting trees with bounded branching.

5. Further results

Although we know that there exists a hyperimmune-free degree with no strong minimal cover, the example of such a degree which we have so far is PA and so certainly cannot be minimal. Of course, what we are really interested in right now is the minimal degrees, and in fact it is possible to achieve a strong result in the opposite direction:

Theorem 5.1. *(Lewis* [AL1]*) Every hyperimmune-free degree which is not FPF is a tree basis and so has a strong minimal cover.*

Corollary 5.1. *Every hyperimmune-free degree bounded by a 1-generic has a strong minimal cover.*

Proof. No 1-generic is FPF and the degrees which are not FPF are downward closed. □

Through an analysis of Gabay's proof [YG] that there exists a minimal degree which is not c.e. traceable, and using also theorem 2.4, it is possible to show that there exist minimal degrees which are hyperimmune-free, not c.e. traceacble and not FPF. Thus the c.e. traceable degrees are actually a proper subclass of those degrees which are a tree basis. Since there are a number of different classes of degrees concerned now perhaps a picture is helpful—the diagram below shows the Turing degrees divided into four basic sections according to whether or not they are FPF and/or hyperimmune-free. The roughly triangular area on the top left (and with no intersection with the area below the horizontal dotted line) is the a.n.c. degrees and the area below the horizontal dotted line depicts the degrees which are c.e. traceable.

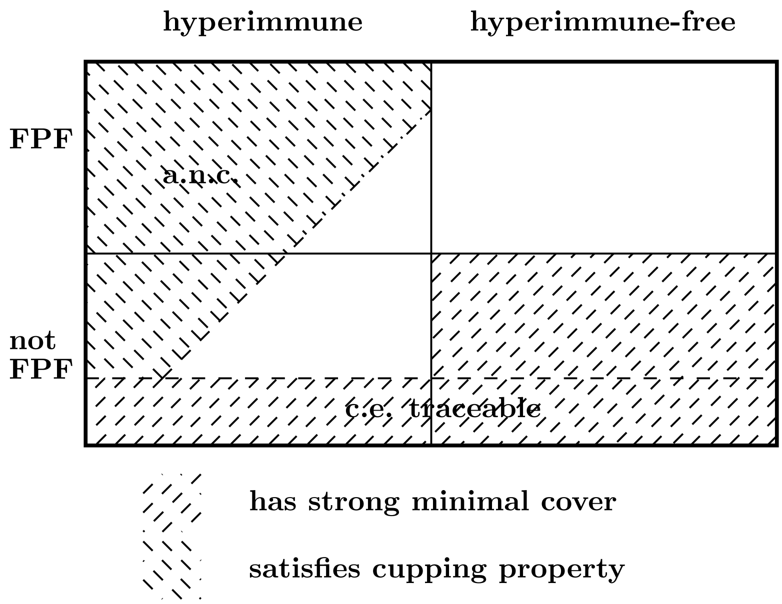

Fig 1. Degree classes with strong minimal cover.

No c.e. traceable degree is FPF and in theorem 5.1 we have also required the condition of not being FPF in order to ensure the existence of a strong minimal cover. On the other hand, all PA degrees satisfy the cupping property and are all FPF. One begins, then, to question whether it might be the case that in fact all FPF degrees satisfy the cupping property, or at least fail to have a strong minimal cover. A positive answer to this question would have been very exciting. Theorem 2.3 would then imply a negative solution to Yates' question and we would also have a complete characterization of the hyperimmune-free degrees with strong minimal cover as those which are not FPF. Unfortunately, however, this question is answered in the negative:

Theorem 5.2. *(Lewis* [AL2]*) There exists a Martin-Löf random degree with strong minimal cover.*

Since Kucera has shown that all Martin-Löf random degrees are FPF theorem 5.2 proves the existence of a FPF degree with strong minimal cover. Since this random degree can also be made hyperimmune-free we therefore have:

Corollary 5.2. *(Lewis* [AL2]*) The hyperimmune-free degrees with strong minimal cover cannot be characterized as those which are not FPF.*

5.1. **Other techniques of minimal degree construction.** The fact that any set of minimal degree lies on delayed splitting trees is not to say that in order to construct a set of minimal degree we need necessarily think in terms of splitting trees. There *are* other ways of constructing a set of minimal degree (even if such methods are not used in practice). One such way involves the use of Cantor-Bendixson rank:

Definition 5.1. *For any $\Lambda \subseteq 2^{<\omega}$ we define $B([\Lambda])$, the Cantor-Bendixson derivative of $[\Lambda]$, to be the set of non-isolated points of $[\Lambda]$ according to the Cantor topology. The iterated Cantor-Bendixson derivative $B^{\alpha}([\Lambda])$ is defined for all ordinals α by the following transfinite recursion. $B^0([\Lambda]) = [\Lambda]$, $B^{\alpha+1}([\Lambda]) = B(B^{\alpha}([\Lambda]))$ and $B^{\lambda}([\Lambda]) = \bigcap_{\alpha<\lambda} B^{\alpha}([\Lambda])$ for any limit ordinal λ. A set A has Cantor-Bendixson rank α if α is the least ordinal such that for some Π^0_1 class $[\Lambda]$, $A \in B^{\alpha}([\Lambda]) - B^{\alpha+1}([\Lambda])$.*

The following theorem was not claimed explicitly by Owen but follows immediately from his results:

Theorem 5.3. *(Owings* [JO]*) If A is hyperimmune-free and of Cantor-Bendixson rank 1 then A is of minimal degree.*

This theorem then combines with a result of Downey's in order to provide us with a technique of minimal degree construction which does not explicitly make use of splitting trees:

Theorem 5.4. *(Downey* [RD]*) There exists a set of hyperimmune-free degree and which is of rank one.*

Of course, the hope might be that using this alternative technique one might be able to avoid complications inherent in the use of the splitting tree technique and thereby produce a negative solution to Yates' question. Such hopes, however, are in vain. Any minimal degree constructed using this particular technique will have a strong minimal cover.

Theorem 5.5. *(Lewis* [AL2]*) If A is of hyperimmune-free degree and is of rank 1 then the degree of A is not FPF and therefore has strong minimal cover.*

We close with a question:

Question 5.1. *Do the degrees with strong minimal cover form a downward closed class?*

If one wishes to characterize the degrees with strong minimal cover then it would seem important to know whether or not these degrees form a downward closed class. Properties such as being c.e. traceable or (being hyperimmune-free and) not being FPF define downward closed classes of degrees. If the degrees with strong minimal cover are not downward closed then certainly no property along those lines is going to be sufficient.

References

[CC] Chong,C.T. (1979) *Generic sets and minimal α-degrees*, Trans.Am.Math.Soc, 254, 157-169.

[BC1] Cooper,S.B. (1973) *Minimal degrees and the jump operator*, J. Symbolic Logic 38, 249-271.

[BC2] Cooper,S.B. (2004) *Computability theory*, Chapman & Hall.

[BC3] Cooper,S.B. (1971) *Degrees of Unsolvability*, PhD Thesis, University of Leicester.

[RD] Downey,R. (1991) *On Π^0_1 classes and their ranked points*, Notre Dame J. of Formal Logic, 32, 499-512.

[DJS] Downey,R., Jockusch,C.G. and Stob,M. (1996) *Array nonrecursive degrees and genericity*, Computability, Enumerability, Unsolvability, vol 224 of Lon.Math.Soc Lecture Note Series, 93-104.

[DY] Downey,R., Yu,L. (2006) *Arithmetical Sacks Forcing*, to appear in the Proceedings of the American Mathematical Society.

[RE1] Epstein,R.L. (1975) *Minimal degrees of unsolvability and the full approximation construction*, Memoirs of the American Mathematical Society, n. 163.

[RE2] Epstein,R.L. (1981) *Initial segments of degrees below* $0'$, Memoirs of the American Mathematical Society, n. 241.

[YE] Ershov,Y.L. (1975) *The uppersemilattice of enumerations of a finite set*, Alg.Log. 14, 258-284.

[YG] Gabay,Y. (2004) *Double jump inversion and strong minimal covers*, PhD thesis, Cornell University.

[SI] Ishmukhametov,S. (1999) *Weak recursive degrees and a problem of Spector*, Recursion Theory and Complexity, Arslanov and Lempp eds., de Gruyter, 81-89.

[AK] Kucera,A. (1985) *Measure, Π^0_1-classes and complete extensions of PA*, Recursion Theory Week (Oberwolfach, 1984), volume 1141 of Lecture Notes in Math. 245-259, Springer, Berlin.

[MK1] Kumabe,M. (1993) *A fixed point free minimal degree*, unpublished.

[MK2] Kumabe,M. (2000) *A 1-generic degree with a strong minimal cover*, Journal of Symbolic Logic, vol. 65, 1395-1442.

[ML] Lerman,M. (1983) *Degrees of unsolvability*, Springer-Verlag.

[AL1] Lewis,A.E.M. (2006) *Π^0_1 classes, strong minimal covers and hyperimmune-free degrees*, submitted.

[AL2] Lewis,A.E.M. (2006) *A random degree with strong minimal cover*, submitted.

[EP] Paliutin,E. (1975) *Addendum to the paper of Ershov*, Alg.Log 14, 284-287.

[PO1] Odifreddi,P.G. (1989) *Classical Recursion Theory Vol.1*, Elsevier.

[PO2] Odifreddi,P.G. (1999) *Classical Recursion Theory Vol.2*, Elsevier.

[JO] Owings,J. (1997) *Rank, join and Cantor singletons*, Archives of Mathematical Logic, 36, 313-320.

[GS] Sacks,G.E. (1963) *Degrees of unsolvability*, Princeton University Press, 2nd ed.

[JS] Shoenfield,J.R. (1966) *A theorem on minimal degrees*, J. Symbolic Logic 31, 539-544.

[RS] Soare,R.I. (1987) *Recursively enumerable sets and degrees*, Springer, New York.

[CS] Spector,C. (1956) *On degrees of recursive unsolvability*, Ann. Math. 64, 581-592.

[TZ] Terwijn,S.A. and Zambella,D. (2001) *Computational randomness and lowness*, J. Symbolic Logic, 66, 1199-1205.

[CY] Yates,C.E.M. (1970) *Initial segments of the degrees of unsolvability, part 2: minimal degrees*, J. Symbolic Logic, 35, 243-266.