1. Introduction

Reducibilities $\leq_{\mathrm{cl}}$ and $\leq_{rK}$ where introduced in [?]. Condition (1.1) of Proposition 1.1 says that for every n and for every c enumerations into $A\upharpoonright_n$ at least one enumeration into $B\upharpoonright_n$ occurs.

Proposition 1.1. *Let A, B be c.e. sets with enumerations satisfying the following property, for some $c \in \mathbb{N}$.*

$$\text{For every } \ell, n \in \mathbb{N}, \text{ if } |A[s+1]\upharpoonright \ell| > |A[s]\upharpoonright \ell|,\ |A[s+1]\upharpoonright \ell| = c\cdot n \text{ then } |B[s+1]\upharpoonright \ell| > |B[s]\upharpoonright \ell|. \tag{1.1}$$

Then $A \leq_{rK} B$.

Proof. It suffices to define a partial computable function $f(\sigma, j)$ such that for each n there exists $j < c$ such that $f(B\upharpoonright_n, j) = A\upharpoonright_n$. The definition of f is as follows: for each σ, j wait for a stage s such that $B\upharpoonright_{|\sigma|}=\sigma$ and the remainder of $|A[s]|$ divided by c is j. Then let $f(\sigma, j) = A[s]\upharpoonright_{|\sigma|}$.

For the verification, first we show that for each n, if $f(B\upharpoonright_n, j)$ is defined and j is the remainder of $|A\upharpoonright_n|$ divided by c then it equals $A\upharpoonright_n$. Indeed, if the definition occurred at stage s and $A[s]\upharpoonright_n$ is not correct, there must occur at least c enumerations into $A\upharpoonright_n$ after stage s. But according to condition (1.1) this means that $B[s]\upharpoonright_n$ is not a prefix of B, a contradiction. Finally for each n it is clear that $f(B\upharpoonright_n, j)$ will be defined if j is the remainder of $|A\upharpoonright_n|$ divided by c. □

Theorem 1.2. *There exists an rK-complete c.e. set. In other words, there exists a c.e. set B such that $W \leq_{rK} B$ for all c.e. sets W.*

Proof. We make use of Proposition 1.1. It is convenient to work with c.e. sets W such that $W(0) = W(1) = 0$, so for the duration of this proof we let W_n be the nth c.e. set satisfying this condition (given the nature of the rK-reducibility this will not affect the uniformity of the reduction constructed). For each n and each ℓ, we ensure that every 2^{n+3} times a number $< \ell$ is enumerated into W_n, a number $< \ell$ is enumerated into B. We do this by considering a number of *boxes*. Each box $\pi_{i,n}$ or $\pi'_{i,n}$ is of size 2^i, which means that we want a maximum of 2^i elements enumerated into it. Whenever a number x with $2^i \leq x < 2^{i+1}$ is enumerated into W_n we enumerate this number into $\pi_{i,n}$ and also into all $\pi'_{j,n}$ such that $j > i$. Every 2^{n+3} times that a number is enumerated into $\pi_{i+1,n} \cup \pi'_{i+1,n}$, we enumerate into B the least number x which has not already been enumerated in with $2^i \leq x < 2^{i+1}$. It is clear that each $\pi_{i,n}$ or $\pi'_{i,n}$ only has a maximum of 2^i elements enumerated into it, and it is also clear that the construction suffices to give the result, so long as there is always an appropriate x with $2^i \leq x < 2^{i+1}$ to enumerate into B. This follows since, for each $i+1$, each of the two boxes $\pi_{i+1,n}$, $\pi'_{i+1,n}$ is twice the size of the interval $[2^i, 2^{i+1})$, so that their union is four times the size. Therefore we add a maximum of $4 \cdot (2^i)/2^{n+3}$ elements to B in the interval $[2^i, 2^{i+1})$ for the sake of each W_n. □

Corollary 1.3. *There exists a $\leq_C$-complete and $\leq_K$-complete c.e. set. Moreover there exists a c.e. set A and a constant c such that $K(W\upharpoonright_n |A\upharpoonright_n) \leq c$ for all n and all c.e. sets W.*

Proof. It is a rather straightforward fact that $\leq_{rK}$ implies (is contained in) $\leq_K$ and $\leq_C$. Also by [?], $\exists c \forall n, K(W\upharpoonright_n |A\upharpoonright_n) \leq c$ is equivalent to $W \leq_{rK} A$. Therefore this is a corollary of Theorem 1.2. □

Theorem 1.4. *Every $\leq_C$-complete and every $\leq_K$-complete c.e. set is also $\leq_{wtt}$-complete.*

Proof. We prove the case for $\leq_K$ as the case for $\leq_C$ is identical. Assume that A is $\leq_K$-complete. Then $W \leq_K A$, where W is a c.e. set that we will construct. Hence, if M is a prefix-free machine, there exists a constant c such that for all n

$$K(W \upharpoonright_n) \leq K_M(A \upharpoonright_n) + c. \tag{1.2}$$

We construct M, W and weak truth table reductions Γ_c, corresponding to the various guesses about the constant c in (1.2). For each c such that (1.2) holds we will have $\Gamma_c^A = \emptyset'$.

Let $\langle i, j\rangle$ be a computable one-one paring function which is monotone in both arguments and such that $\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} 2^{-\langle i,j\rangle} < 1$. Consider a partition of $\mathbb{N}$ into consecutive intervals I_k such that I_k contains 2^{k+c} numbers that are smaller than all numbers in I_{k+1}. The possible enumeration of n into $\emptyset'$ will be coded in to the intervals $I_{\langle n,c\rangle}, c \in \mathbb{N}$ of W. This coding will propagate into A, in the cases where the constants c satisfy (1.2). Let k_n^c be the least number which is larger than all numbers in the interval $I_{\langle n,c\rangle}$. We say that $\langle n, c\rangle$ requires attention at stage $s+1$ if $\langle n, c\rangle \leq s$ and the following conditions are met:

(a) $n \in \emptyset'[s]$;
(b) If s_0 is the first stage where $n \in \emptyset'[s_0]$ then $A[s_0] \upharpoonright_{k_n^c} \subset A[s]$;
(c) $K(W \upharpoonright_{k_n^c})[s] \leq K_M(A \upharpoonright_{k_n^c})[s] + c$.

Without loss of generality we may assume that a number n may only be enumerated into $\emptyset'$ at a stage which is larger than $\langle n, 0\rangle$.

Construction of W and M. At stage $s+1$, for each $\langle n, c\rangle \leq s$ that requires attention do the following. If $n \in \emptyset'[s+1] - \emptyset'[s]$ then enumerate a description of $A \upharpoonright_{k_n^c}$ of length $\langle n, c\rangle$. Otherwise enumerate the largest element of $\mathbb{N} - W$ in the interval $I_{\langle n,c\rangle}$ into W.

Verification. For each $(n, c) \in \mathbb{N}\times\mathbb{N}$ at most one M description of length $\langle n, c\rangle$ is requested. Hence M is a prefix-free machine. First note that at least one number in each $I_{\langle n,c\rangle}$ will remain outside W (so that all requested enumerations into W are possible). Indeed, otherwise the universal machine would have to produce descriptions of total weight 1. Next we show that if (1.2) holds for some c, then whenever n is enumerated into $\emptyset'$ at some stage $s_1 > \langle n, c\rangle$ there is always a stage $s_2 > s_1$ such that $A[s_1] \upharpoonright_{k_n^c} \not\subset A[s_2]$ (this provides a weak truth table reduction Γ_c of $\emptyset'$ to A). Indeed, assuming that this was not the case for some n, c implies that $\langle n, c\rangle$ requires and receives attention $2^{\langle n,c\rangle+c}$ times. On the assumption that $A[s_1] \upharpoonright_{k_n^c} \not\subset A$ we have $K_M(A \upharpoonright_{k_n^c}) = \langle n, c\rangle$ and the recursive action of the construction on $\langle n, c\rangle$ in conjunction with (1.2) would force the universal machine to enumerate at $2^{\langle n,c\rangle+c}$ descriptions of length $\langle n, c\rangle + c$, which is a contradiction. □

On the other hand it is not hard to see that there are many-one complete c.e. sets which are not $\leq_K$-complete.

Theorem 1.5. *Every c.e. set can be split into two c.e. sets of the same K-degree. In other words, if A is a c.e. set then there exist c.e. sets A_0, A_1 such that $A_0 \cup A_1 = A$, $A_0 \cap A_1 = \emptyset$ and $A \equiv_K A_0 \equiv_K A_1$.*

Proof. We fix a computable enumeration of A and define the splitting A_0, A_1 as in the statement of the theorem. It suffices to construct a prefix-free machine M

such that the following requirements are met for all n:

$$K_M(A\upharpoonright_n)\leq K(A_i\upharpoonright_n) \tag{1.3}$$

Without loss of generality we may assume that enumerations into A happen only at odd stages and that at each stage at most one such enumeration takes place. Also we may fix a universal prefix-free machine U, which is used for the definition of K and which has weight less than $1/4$. At each odd stage $s+1$ we will be concerned with the weights

$$w_i(n)[s]=\sum_{n<k\leq s}2^{-K(A_i\upharpoonright_k)[s]}.$$

Construction. At each odd stage $s+1$, if n enters A at this stage let j be (the least number) such that $w_j(n)[s]\leq w_{1-j}(n)[s]$. Then enumerate n into A_{1-j}. At each even stage $s+1$ and each $n\leq s$ such that $K_M(A\upharpoonright_n)[s]>K(A_i\upharpoonright_n)[s]$ enumerate an M-description of $A[s]\upharpoonright_n$ of length $K(A_i)[s]$.
Verification. By the construction, it suffices to show that the requests that we enumerate for M during the construction have weight at most 1. Each request enumerated in M at stage $s+1$ is triggered by $K_M(A\upharpoonright_n)[s]>K(A_i\upharpoonright_n)[s]$ for some n and some $i=0,1$. In this way we may divide M into two machines M_0,M_1 corresponding to A_0,A_1 respectively. We show that the weight of the M_0-requests is at most $1/2$. A symmetric argument shows that the weight of the M_1-requests is also at most $1/2$, so this will conclude the proof.

Each M_0-request at stage $s+1$ is triggered by $K_M(A\upharpoonright_n)[s]>K(A_0\upharpoonright_n)[s]$ for some n, i.e. by the (leftmost) shortest $U[s]$-description τ of $A_0\upharpoonright_n[s]$. In this case we say that τ becomes *used*. Each U-description becomes used once, and then it remains used for the rest of the construction. Let us say that an M_0-request is *primary* if it corresponds to an *unused* U-description, in the way we defined above. If an M_0 request is not primary, we call it *secondary*. Clearly the weight of the primary M_0-requests is bounded by the weight of the universal machine (which determines $K(A_i\upharpoonright_n)$ and its approximations). Since the latter is less than $1/4$, it suffices to show that the same holds for the weight of the secondary M_0-requests.

Note that if at odd stage s no number is enumerated into A then any M_0-requests at stage $s+1$ will be primary. Moreover the same holds if a number is enumerated in A_0 at stage s. Hence if at $s+1$ a secondary M_0-request is enumerated, it is necessarily the case that some number (smaller than the lengths of the strings for which the secondary requests were issued) was enumerated in A_1 at stage s. We show that for every increase in the weight of the secondary M_0 request we can count an equal (or even larger) increase in the weight of the universal machine U. Indeed, if at stage $s+1$ some secondary M_0 requests are enumerated, a number n which is smaller than all the lengths of these secondary requests must have entered A_1 at stage s. According to the construction, this means that $w_0(n)[s-1]\leq w_1(n)[s-1]$. Hence we can count weight $w_1(n)[s-1]$ in the domain of U, which corresponds to descriptions of initial segments of $A[s-1]$ of lengths $>n$. Since $A(n)[t]\neq A(n)[s-1]$ for all $t\geq s$ this weight in the domain of U will not be counted twice. It follows that the weight of the secondary M_0-requests is also bounded by $1/4$. Hence the weight of the M_0-requests is bounded by $1/4+1/4=1/2$. This (and the entirely symmetric argument for M_1) shows that the weight of the M-requests is bounded by 1. □

Question 1. *Are the K-complete c.e. sets also tt-complete? Can they be?*

Question 2. *Is every sequence rK-reducible to a random sequence?*